UK Maths Trust

Intermediate Mathematical Olympiad

Cayley paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

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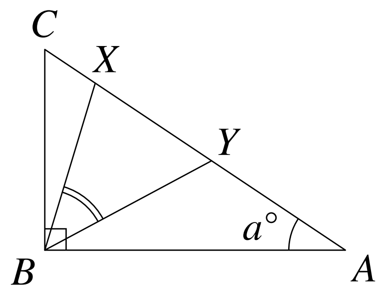
1. A triangle, ABC, has a right angle at B and $\angle A = a^\circ$. X lies on AC such that $AX = AB$. Y lies on AC such that $CY = CB$. Prove that $\angle XBY$ has the same size independently of the other angles of the triangle and find the size of that angle.



Solution

As the sum of internal angles in a triangle is 180°, we have $\angle C = (90 - a)^\circ$.

The triangle ABX is isosceles (because $AX = AB$), hence

$$\angle AXB = \angle ABX = \frac{180^\circ - \angle A}{2} = \left(90 - \frac{a}{2}\right)^\circ.$$

Similarly, the triangle CBY is isosceles (because $CY = YB$), hence

$$\angle BYC = \angle CBY = \frac{180^\circ - \angle C}{2} = \left(90 - \frac{90 - a}{2}\right)^\circ = \left(45 + \frac{a}{2}\right)^\circ.$$

Consider now the sum of internal angles of the triangle BXY:

$$180^\circ = \angle YXB + \angle XBY + \angle BYX = \angle AXB + \angle BYX + \angle BYC$$

$$\angle BYX = 180^\circ - \angle AXB - \angle BYC = \left(180 - \left(90 - \frac{a}{2}\right) - \left(45 + \frac{a}{2}\right)\right)^\circ = 45^\circ.$$

This shows that $\angle XBY = 45^\circ$, independently of the other angles of the triangle.

2. The Intermediate Maths Challenge has 25 questions with the following scoring rules:
5 marks are awarded for each correct answer to Questions 1-15;
6 marks are awarded for each correct answer to Questions 16-25;
Each incorrect answer to Questions 16-20 loses 1 mark;
Each incorrect answer to Questions 21-25 loses 2 marks.
Where no answer is given 0 marks are scored.
Fiona scored 80 marks in total. What possible answers are there to the number of questions Fiona answered correctly?

Solution

As each question is worth at most 6 marks, it is not possible to get more than $13 \times 6 = 78$ marks by correctly answering 13 (or fewer) questions.

If Fiona answered correctly 18 questions, the lowest number of marks she could get for her correct answers would be $15 \times 5 + 3 \times 6 = 93$ and the highest number of marks she could lose in the remaining seven questions would be $5 \times 2 + 2 \times 1 = 12$. Hence by answering 18 (or more) questions correctly, Fiona would always score at least $93 - 12 = 81$ marks.

Fiona could have answered correctly 14, 15, 16 and 17 questions, for example:

- 14: correct answers to questions 12 – 25 ($4 \times 5 + 10 \times 6$),
- 15: correct answers to questions 6 – 20 ($10 \times 5 + 5 \times 6$),
- 16: correct answers to questions 1 – 16 and an incorrect answer to question 17,
- 17: correct answers to questions 1 – 17 and incorrect answers to questions 20 – 23.

3. An integer $N < 2024$ is divisible by 39 times the sum of its digits. Find all possibilities for N.

Solution

As N is a multiple of 39, it is also a multiple of 3.
Hence, the sum of its digits is a multiple of 3.
Hence, N is a multiple of $39 \times 3 = 117$, and in particular also a multiple of 9 (as $117 = 9 \times 13$).
Hence, the sum of its digits is a multiple of 9.
Hence, N is a multiple of $39 \times 9 = 351$.

The only multiples of 351, smaller than 2024, are: 351, 702, 1053, 1404, and 1755.
The first four of them each have their digit sum of 9, hence they are multiples of 39 times the sum of its digits (which is 351). In case of 1755, its digit sum is 18. As an odd number, 1755, is not divisible by 18.

Hence, the only positive integers $N < 2024$, divisible by 39 times the sum of its digits, are 351, 702, 1053, and 1404.

4. When written in ascending order, the nine internal angles from three particular triangles form a sequence where the difference between any adjacent pair of numbers in the sequence is a constant d. One of the angles measures $42°$. Find all possible values of the size of the largest of the nine angles.

Solution

Let m be the median of the nine angles. We can express the nine angles (in degrees) as

$$m - 4d,\ m - 3d,\ m - 2d,\ m - d,\ m,\ m + d,\ m + 2d,\ m + 3d,\ m + 4d.$$

As the sum of internal angles in each triangle is $180°$, these nine numbers add up to $3 \times 180 = 540$. Hence, $9m = 540$, so $m = 60$.

We know that one of the numbers $m - 4d, m - 3d, m - 2d, m - d$ is equal to 42. Consider each of these four cases in turn:

- Case 1 ($m - 4d = 42$). This yields $d = 4.5$, and angle measures of

  $$42°,\ 46.5°,\ 51°,\ 55.5°,\ 60°,\ 64.5°,\ 69°,\ 73.5°,\ 78°.$$

  We can easily construct three triangles, with their angles, respectively, $(42°, 64.5°, 73.5°)$, $(46.5°, 55.5°, 78°)$ and $(55.5°, 60°, 64.5°)$.

- Case 2 ($m - 3d = 42$). This yields $d = 6$, and angle measures of

  $$36°,\ 42°,\ 48°,\ 54°,\ 60°,\ 66°,\ 72°,\ 78°,\ 84°.$$

  We can easily construct three triangles, with their angles, respectively, $(36°, 66°, 78°)$, $(42°, 54°, 84°)$ and $(48°, 60°, 72°)$.

- Case 3 ($m - 2d = 42$). This yields $d = 9$, and angle measures of

  $$24°,\ 33°,\ 42°,\ 51°,\ 60°,\ 69°,\ 78°,\ 87°,\ 96°.$$

  We can easily construct three triangles, with their angles, respectively, $(24°, 69°, 87°)$, $(33°, 51°, 96°)$ and $(42°, 60°, 78°)$.

- Case 4 ($m - d = 42$). This yields $d = 18$, which is impossible, as $m - 4d = -12 < 0$.

In conclusion, the only possible values for the size of the largest angle are $78°$, $84°$ and $96°$.

5. A large number of people arrange themselves into groups of 2, 6 or 10 people. The mean size of a group is 5. However, when each person is asked how many other people are in their group (excluding themselves), the mean of their answers is 7. Prove that there are no groups of 6 people.

SOLUTION

Let x be the number of groups of two people, let y be the number of groups of six, and let z be the number of groups of ten. The total number of groups is $x + y + z$ and the total number of people is $2x + 6y + 10z$. We can write the mean size of a group condition as

$$2x + 6y + 10z = 5(x + y + z) \tag{1}$$

When asked how many other people are in their group:

- The $2x$ people from x groups of two reply '1',
- The $6y$ people from y groups of six reply '5',
- The $10z$ people from z groups of ten reply '9'.

We can write the sum of their $2x + 6y + 10z$ answers as $2x + 30y + 90z$. Hence,

$$2x + 30y + 90z = 7(2x + 6y + 10z) \tag{2}$$

Expanding the brackets and collecting like terms in equation (1) and (2), we get

$$3x - y - 5z = 0 \tag{3}$$

$$12x + 12y - 20z = 0 \tag{4}$$

By multiplying equation (3) by four, and subtracting the resulting equation from (4), we have $16y = 0$, hence $y = 0$. This proves there are no groups of six people.

6. Into each row of a 9×9 grid, Nigel writes the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 in order, starting at one of the digits and returning to 1 after 9: for example, one row might contain 7, 8, 9, 1, 2, 3, 4, 5, 6. The grid is *gorgeous* if each nine-digit number read along a row or column or along the diagonal from the top-left corner to the bottom-right corner or the diagonal from the bottom-left corner to the top-right corner is divisible by 9. How many of the 9^9 possible grids are *gorgeous*?

Solution

Firstly, note that each row of the grid contains numbers $1 - 9$ in some order, so their sum is always $1 + 2 + 3 + \ldots + 9 = 45$, which is a multiple of 9.

Then, notice that adding 9 to any cells in the grid does not change divisibility by 9 of any row, column, and diagonal sums. Hence, instead of writing rows as described in the question:

5	6	7	8	9	1	2	3	4
3	4	5	6	7	8	9	1	2
9	1	2	3	4	5	6	7	8

Nigel will start filling each row from the left with any number between 1 and 9, and continue writing consecutive integers throughout the row. The example above will change to:

5	6	7	8	9	10	11	12	13
3	4	5	6	7	8	9	10	11
9	10	11	12	13	14	15	16	17

Consider a situation where Nigel completed the first eight rows of the grid in any of the 9^8 possible ways (each way corresponds to choosing the values of $1 \leq a_1, a_2, \ldots, a_8 \leq 9$):

a_1	$a_1 + 1$	$a_1 + 2$	$a_1 + 3$	$a_1 + 4$	$a_1 + 5$	$a_1 + 6$	$a_1 + 7$	$a_1 + 8$
a_2	$a_2 + 1$	$a_2 + 2$	$a_2 + 3$	$a_2 + 4$	$a_2 + 5$	$a_2 + 6$	$a_2 + 7$	$a_2 + 8$
				$\cdots$				
a_8	$a_8 + 1$	$a_8 + 2$	$a_8 + 3$	$a_8 + 4$	$a_8 + 5$	$a_8 + 6$	$a_8 + 7$	$a_8 + 8$

Let $1 \leq a_9 \leq 9$ be such a number that $a_1 + a_2 + \ldots + a_8 + a_9$ is a multiple of 9. There is exactly one value satisfying this condition, equal to $9 - r$, where r is the remainder when $a_1 + a_2 + \ldots + a_8$ is divided by 9.

The only number Nigel can potentially use to start the ninth row is a_9, as any other number will make the sum of the first column not divisible by 9. Note that the numbers in each column are larger by one than the corresponding numbers from the previous column. Hence, the column sums increase by 9, going from left to right. This means that if the sum of the first column is divisible by 9, so are the sums of all the other columns.

Consider now the sums on the two main diagonals:

- $a_1 + (a_2 + 1) + (a_3 + 2) + \ldots + (a_9 + 8) = (a_1 + a_2 + \ldots + a_9) + (1 + 2 + \ldots + 7 + 8) = (a_1 + a_2 + \ldots + a_9) + 36$, which is a multiple of 9.

- $(a_1 + 8) + (a_2 + 7) + \ldots + (a_8 + 1) + a_9 = (a_1 + a_2 + \ldots + a_9) + (8 + 7 + \ldots + 2 + 1) = (a_1 + a_2 + \ldots + a_9) + 36$, which is a multiple of 9.

This proves that for each of the 9^8 choices of $(a_1, a_2, \ldots, a_8)$, there exists exactly one value of a_9 which completes a *gorgeous* grid. Hence, there are exactly 9^8 possible *gorgeous* grids.